Underpinnings
2. Representing Data With Graphs

Discussion—Graphical Representation of Data

A graph provides a compact, pictorial presentation of data. Graphs are sometimes used to present qualitative relationships between quantities with a picture as, for example, in a bar graph or pie graph. In experimental science, the most important use of graphs is not only to display the relationships between measured quantities as clearly as possible, but to make the presentation in a form that allows maximum quantitative assessment of an experiment. An experiment usually consists of a series of measurements of a pair of quantities that are assumed to be related to each other. One of the quantities is changed, at will, by the experimenter (this quantity is called the independent variable) and the second quantity (called the dependent variable) will change automatically in a regular manner. In mathematical language, we say that the dependent variable is a "function" of the independent variable.

Each data point on a graph consists of a pair of numbers which define one particular value of the independent variable along with the corresponding value of the dependent variable. A data point is represented by the notation (x, y) where x is the value of the independent variable and is the number stated first inside the parentheses and y is the dependent variable. Thus, (2.0, 4.6) corresponds to a data point in which the independent variable has value x = 2.0 and the corresponding value of the dependent variable is y = 4.6. A data point is represented on a graph by its location with respect to two perpendicular lines called axes—the horizontal axis represents the independent variable and the vertical axis represents the dependent variable. Thus, the position of the data point (2.0, 4.6) is determined by finding 2.0 on the horizontal scale, then moving straight up, parallel to the vertical axis until even with 4.6 on the vertical scale.

When a graph is plotted, it is necessary to establish scales for the axes. Scales should be chosen so that all of your data fits on the graph and the scales should be easy to read. This can be accomplished by numbering the grid lines on the graph axes so that it is easy to estimate points that lie between two grid lines. This is illustrated by considering the two cases shown to the right—in which case is it easier to read the value indicated by the arrow?

It is important to remember that data measurements have inherent uncertainty so that, when represented on a graph, the data points will not define a perfectly smooth pattern. If one tries to draw a smooth curve that represents the trend of the data, the smooth curve that is drawn will not go through all of the data points, i.e., the points will "scatter" on either side of the curve. In drawing a graph that represents the data, the usual practice is to assume that half the experimental measurements are too high and half are too low and we draw a smooth "best fit" line or curve through the middle of the scattered points that represents the trend shown by the data. In drawing a curve on a graph, the data points are never connected by a connect-the-dots zigzag line as shown below. This would imply that the independent and dependent variables are related to one another in a very irregular way. It would also imply that the data points were exact and had no uncertainty.
**Discussion continued—**

A graph gives information not only about each point, but also about relations between points. For example, the figure to the right shows two data points on a graph of mass versus volume. The horizontal length, \( a \), represents a number of cubic centimeters—specifically, it is the number of cubic centimeters (7) that would change point 1 to point 2. A volume change of 7 cm\(^3\) is accompanied by an increase of 14 g in the mass, as indicated by the vertical length, \( b \). Note that the relation between points on a graph is defined by the horizontal and vertical lengths \( a \) and \( b \), not the straight line distance, \( c \), which is shown in the figure as a dotted line. The length of this line represents neither a number of grams nor a number of cubic centimeters and has no physical meaning. The vertical length between the two points is the rise and the horizontal length is the run. The ratio of rise to run is constant for straight-line graphs and it does not matter which points are used to compute this ratio. For straight-line graphs, the ratio of rise to run is called the slope—if the graph is a straight-line, the slope is constant. Conversely, if the slope is not constant, the graph is a curve, not a straight line.

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**Part I - Graphical Analysis of Mass vs. Volume**

1. Record the mass of a 100 mL graduated cylinder. Using a medicine dropper carefully place 20 mL of water into the graduated cylinder. (When finding the volume of a liquid in a graduated cylinder, read the marking at the bottom of the liquid meniscus. The meniscus is the curvature of the liquid due to the adhesion of the liquid to the walls of the container). Measure the mass of the graduated cylinder containing the 20 mL of water and, by subtraction, determine the mass of the water. Record your data in the following table.

\[
\begin{array}{|c|c|c|}
\hline
\text{Volume of liquid (mL)} & \text{Mass of liquid + graduated cylinder (gram)} & \text{Mass of liquid (gram)} \\
\hline
0 & & \\
20 & & \\
40 & & \\
60 & & \\
80 & & \\
100 & & \\
\hline
\end{array}
\]

Without emptying your graduated cylinder, add another 20 mL of water using the medicine dropper and determine the mass of the 40 mL of water now in the graduated cylinder. Continue this procedure, 20 mL at a time until you have 100 mL of water in your graduated cylinder.

A. What are the independent and dependent variables in this experiment? Explain.

2. Plot your data on a graph of mass versus volume on the graph paper provided. (Choose your scale for the axes carefully and label the axes with the name and units of the variable that it represents.)

A. Does the trend of your data points define any particular pattern? What pattern is discerned in your data?

UP-2.2
B. Your data should look like a straight line. Is it? If not, check your measurements and your scales assigned to the axes on your graph. Determine the slope of your line. On your graph, show how you determined the slope.

C. What are the units of the slope? Give an interpretation for the slope, i.e., what does the slope tell you about the liquid used in your experiment (in this case water)?

**Part II - Keeping Things in Proportion**

**Discussion—Direct and Inverse Proportions**

When two variables, x and y, are related by the equation \( y = mx + b \), a mathematician would say that x and y share a **linear** relationship and, as you know, a graph of y vs. x would be a straight line. The slope, m, is a constant and measures how steep the line is and b, the intercept, defines the point at which the line crosses the y-axis. A **direct proportion** is a special case of a linear relation that arises when \( b = 0 \), which means that a graph of y vs. x will be a straight line that goes through the origin (0, 0). *(Note: Not all linear relations are direct proportions—most are not. However, all direct proportions ARE linear relations.)* For a direct proportion, with \( b = 0 \), the relationship between y and x becomes \( y = mx \), and, since m is a constant, it will be convenient to write:

\[
y = (\text{constant}) \cdot x \quad \text{(Direct Proportion)}
\]

In this equation, the *constant* is called the **constant of proportionality**. If we divide both sides of this equation by x, it becomes:

\[
\frac{y}{x} = \text{constant} \quad \text{(Direct Proportion)}
\]

which gives us a basis for another way to define a direct proportion:

*If two quantities (y and x) are directly proportional, then the ratio of these quantities is always the same (i.e., it is constant).*

One very interesting thing about a direct proportion is that changes in the two proportional variables must occur at the same rate. To understand this, consider that if x doubles (from 2 to 4, 17 to 34, 0.3 to 0.6—it does not matter), then the ratio \( y/x \) now has a denominator that is twice as big as before. Since the ratio \( y/x \) must remain unchanged, the numerator \( y \) must also double. Similarly, if x triples, then y must triple, if x is halved, then y must also be reduced by half, and so on. These examples demonstrate that, for two variables that are proportional, when one of the variables is changed, the other variable must change at the same rate.

Now consider the possibility that \( y \) is **proportional to the inverse of x** (or, as is more commonly stated, \( y \) is inversely proportional to \( x \)) which would be written as

\[
y = \text{constant} \cdot \left( \frac{1}{x} \right) \quad \text{(Inverse Proportion)}
\]

Forming the ratio of \( y \) and \( 1/x \) gives,

\[
\frac{y}{\left( \frac{1}{x} \right)} = \text{constant} \quad \Rightarrow \quad y \cdot \left( \frac{x}{1} \right) = \text{constant}
\]

and we see that the condition for inverse proportionality can also be written as

\[
yx = \text{constant} \quad \text{(Inverse Proportion)}
\]

**UP-2.3**
We see that, unlike direct proportions—where dividing \( y \) by \( x \) always gives the same answer—in an inverse proportion, multiplying \( y \) by \( x \) will always give the same answer. As an example, consider that \( y \) and \( x \) are inversely proportional and that \( x = 2, \ y = 16 \)—i.e., \( xy = (2)(16) = 32 \). No matter how \( x \) changes, \( y \) must change so that the product of \( x \) and \( y \) will still be 32. Thus, if \( x \) is doubled from 2 to 4, to find the effect on \( y \), we ask: “what must be the value of \( y \) when, multiplied by \( x = 4 \), will give 32 as the result”? The answer is, of course, 8. Thus, when \( x \) is doubled (from 2 to 4) \( y \) is halved (from 16 to 8).

As noted above, the form of the equation \( y = (\text{constant}) \cdot x \) is linear and a graph of \( y \) vs. \( x \) will be a straight line with the slope equal to the proportionality constant. On the other hand, the characteristic equation of an inverse proportion—\( xy = \text{constant} \)—is NOT linear and a graph of \( y \) vs. \( x \) in this case will be a curve. However, in an inverse proportion, since \( y \) is proportional to the inverse of \( x \), we can think of \( 1/x \) as the variable and a graph of \( y \) vs. \( 1/x \) (rather than \( y \) vs. \( x \)) will produce a straight line, also with the slope equal to the proportionality constant.

1. A. Using at least five circular objects of different sizes, as accurately as you can, measure the circumference and diameter of each object. Note that you are to measure the circumference and diameter and NOT calculate by using some formula. (For each object, it may be a good idea to measure the circumference and diameter more than once and use an average value to minimize errors.) Record your data in the table below.

<table>
<thead>
<tr>
<th>Object</th>
<th>Circumference (( C )) (cm)</th>
<th>Diameter (( d )) (cm)</th>
<th>Ratio (( C/d ))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

B. Does it appear from the data in the table that \( C \) and \( d \) are proportional? How can you tell?

2. Plot your data on a graph which has circumference as the vertical axis and diameter as the horizontal axis. Define your scales so that your graph fills most of the graph paper.

A. Draw a smooth line through the middle of your data.

(Do this by holding a taut piece of thin string or a stretched rubber band on your graph and adjusting the orientation of the string so that there are some data points above the string and some below. The "best fit" is approximated by the orientation for which the total deviation from the line of points above is about the same as the total deviation from the line of points below. Once the string is oriented at what appears to be the "best" orientation to balance the deviations, have your partner mark with a pencil two points that the string or rubber band passes through (i.e., two points that lie on your "eyeballed" best line). Remove the string and, using a ruler, connect these two points to draw the best line through the data. This method of "eyeballing" the best line can be done with a ruler but, unless the ruler is transparent, it is a bit more difficult this way because the ruler hides half the data points—either above or below the ruler edge—so that it is difficult to visually "average out" the deviations.)

B. The "best fit" to your data should be a straight line. Is it? If not, check your measurements and your scales assigned to the axes on your graph.
3. A. In the discussion above, the following statement is made:

"Not all linear relations are direct proportions—most are not. However, all direct proportions ARE linear relations."

(i) Besides being linear and graphing as a straight line, what condition distinguishes a direct proportion from other straight lines?

(ii) Does your graph in #2 indicate that the circumference and diameter are proportional? Explain.

B. Determine the slope of the best-fit line and write a mathematical expression relating C and d.

C. Compare the slope found in 3B to the ratios in the table in 1A. Are these numbers similar? Do they look like any number that you have seen before associated with circles? If so, what number?

D. What does the slope tell you about circles, any and all circles?

F. Using the information from your graph and using what you know about the equation for a straight line, write a mathematical expression relating C and d.

G. (i) Given a certain circle, for every one centimeter larger you make the diameter, how much larger must the circumference be? Is that true for any given circle?

(ii) If a circle is enlarged so that its diameter is quadrupled, how much bigger is the circumference of the enlarged circle? Explain.

UP-2.5
Part III - Height of Liquid in a Container vs. Volume

1. Given the container shown to the right, sketch a graph with height of water level above the table on the vertical axis and volume of water in the container on the horizontal axis. (The purpose of this activity is to think about identifying trends so the graph need only be qualitative showing approximately the way the height rises as the volume is changed, i.e., as water is poured into the container.)

   A. Sketch your graph and discuss your ideas with your partner.

2. On the same sketch that you made above for container (A), sketch another graph of height vs. volume for container (B) shown to the right.

   A. Discuss your graph with your partner and explain how your two graphs for containers (A) and (B) differ. Explain your thinking.

3. Now consider the conical flask (container (C)) shown to the right. Sketch a graph of height vs. volume for container (C).

   A. Discuss your graph with your partner and, again, explain your thinking.

   B. Can you think of a way to check your predicted graph by measurement?

4. Your group will be given a container shaped like one of the three shapes above.

   Indicate which container your group has been assigned

   (A) Large diameter cylinder  (B) Skinny Cylinder  (C) Conical Flask

   Using a graduated cylinder, progressively add measured volumes of water in 10 mL increments to the container and measure the height of the water from the table top. Record your data in the following table:

   UP-2.6
**Data Table:**

<table>
<thead>
<tr>
<th>Total Volume of water in container (mL)</th>
<th>Height of surface above table top (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Extend your data table until you have filled the container.

A. Plot your data on a piece of graph paper provided. (Each student should do this so that you each have a copy of your results for yourself.) (Note that this time you are *plotting* a graph with real data rather than drawing an approximate *sketch*.

If you are working with either of the two cylindrical containers, determine the slope of your graph.

B. Your instructor will have at least two teams for each container shape draw a sketch of the graph that represents their data onto the chalkboard. This graph drawn on the chalkboard need only be an approximate sketch showing the general features of the graph that was plotted in part A.

If you are asked to sketch your graph on the board and, if you are working with either of the two cylindrical containers, also write the slope of your graph on the board.

C. How do the graphs displayed on the chalkboard compare to your predicted graphs in parts 1-3? Do the plotted graphs confirm your predictions? If not, go back and reexamine your thinking used to make your predictions.

E. Look closely at the conical flask shaped like container (C). Are the volume marks on the flask equally spaced? Explain why you think that they are spaced the way they are.
Discussion—Curved Graphs...

Imagine a graph that is curved, rather than a straight line. Since the slope is defined for, and has definite meaning for, a straight line, does the word "slope" have any meaning when we are referring to a curve? For example, does it have meaning to ask "what is the slope of the curve shown to the right"? To answer this question, consider point "A" indicated by the dot at the top of the hill in the figure. Imagine that we draw a straight line that just touches the curved graph only at point A, as shown. Mathematicians call a line that touches a curve at only one point a tangent line or, more simply, a tangent. Since the tangent that touches the curve at point A is a straight line, we can define its slope in the usual way, i.e., at point A, the slope of the tangent line is 0. Frequently, the specific reference to the tangent is dropped and we refer to "the slope of the curve at point so-and-so", where it is understood that we really mean "the slope of the tangent line touching the curve at point so-and-so".

Now, imagine two additional points, B and C, also on our curved graph, as shown to the right. To find the slope of the curve at point B, we would draw the tangent line at point B and determine the slope of this line (labeled "tangent B" in the figure). We can do a similar thing at point C ("tangent C"'). The slope of tangent B is greater than the slope of tangent C and it should be clear that the curve is rising more steeply at B than at C. The slope of a curve at a particular point determines how rapidly the graph is rising vertically for a given change in the horizontal direction, i.e., it is a measure how steep the curve is at that point.

This can be understood by imagining yourself to be standing on a long board placed at different positions on a steep hill as pictured to the right. The board on which you stand at any point will lie in the direction of the tangent line. At what location do you think the hill would feel steepest?

5. Shown below are five containers of different shapes. (For the drinking glass, the bottom section shown in black is solid glass.) Below the containers are six graphs of height vs. volume. Can you match the correct graph with each bottle? For the remaining graph, sketch what you think the bottle looks like.
6. On the basis of your understanding of the way the height changes with volume, consider the following questions:

(i) What does it mean if the graph is a straight line?

(ii) What does it mean if the first point is up on the vertical axis?

(iii) What would it mean if the first point on the graph was out on the horizontal axis, i.e. looked like that shown to the right?

(iv) If the slope of the graph is getting steeper and steeper, what would that tell you about the shape of the container?

(v) If the slope of the graph is becomes less and less steep, what would that tell you about the shape of the container?

(vi) What is the meaning of a single point on the graph? What does it tell you about the shape of the container?

(viii) The operation of a thermometer depends upon the fact that when the liquid in the thermometer warms, it expands and the volume increases. Examine a thermometer. Approximately how many times as wide as the narrow tube is the bulb? Using evidence from above, explain why it is easier to observe the change in volume using a narrow tube rather than a wide tube.

6. In Part 4, you progressively added measured volumes of water to a container and, for each new volume, you measured the height of the water and plotted a graph of height vs. volume for the container. Now, imagine that we had a way to measure the height of the water “on the fly”. Then, as the water fills the container, we can measure the height instantly and record the time, measure the height a few seconds later and again record the time, and so on. This way we would collect data for a graph of height vs. time, rather than height vs. volume.

A. After reading the above paragraph a student reasons that, if water flows into a container from a faucet at a constant rate, a graph of height vs. time for cylinders (A) and (B) would look like the graphs shown to the right—i.e., the graphs would have the same general shape as the height vs. volume graphs studied previously. (Note: the axes are height vs. time, not height vs. volume.)

The student is correct. Explain why this is true.

B. What is the interpretation of the slope of the lines in 6A? Explain.
C. Shown below is a sketch of a graph of water height vs. time for the conical flask (container (C)), as it was filled at a constant rate from the faucet, as described in 6A.

As was the case for cylinders A and B, the height vs. time graph has the same general appearance as the height vs. volume graph. Indicated on the graph are three points—A, B, and C—occurring at times $t_A$, $t_B$, and $t_C$, respectively. We will want to draw the tangent lines at points A, B and C but, first consider the following discussion...

**Discussion...**

To visually estimate the orientation of a tangent to a curve, we make use of the fact that, for a small enough “piece”, even a curve will appear to be a straight line. (When you stand on the Earth, it appears flat, even though you are on the surface of a very large sphere.) At the point where it is desired to draw a tangent, *imagine* a very tiny line segment oriented in the direction of the curve. Extend this line segment in both directions to form the tangent line. This is pictured in the figure below for two points, P and S, on a curve. The thickness of the short segment of the graph is exaggerated to illustrate how the imagined segments at P and S are oriented in the direction of the curve.

The short line segments in the diagram are only intended as a mental guide to determine the orientation of the tangent line. It is not necessary to draw an actual line segment—just draw the tangent line touching the curve at the point of interest in the direction of the imagined line segment.

C. (i) With the above discussion in mind, draw approximate tangents to the curve above at each of the points A, B, and C.

(ii) How does the slope of the tangents drawn in C(i) relate to steepness of the curve at points A, B, and C?

(iii) How does the slope of the tangents drawn in C(i) relate to the rate at which the water height is increasing as time advances from point A to B to C? Explain.